

## THE MINIMAL CARDINALITY WHERE THE REZNICHENKO PROPERTY FAILS

BY

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### ABSTRACT

A topological space  $X$  has the Fréchet–Urysohn property if for each subset  $A$  of  $X$  and each element  $x$  in  $\overline{A}$ , there exists a countable sequence of elements of  $A$  which converges to  $x$ . Reznichenko introduced a natural generalization of this property, where the converging sequence of elements is replaced by a sequence of disjoint finite sets which eventually intersect each neighborhood of  $x$ . In [5], Kočinac and Scheepers conjecture:

The minimal cardinality of a set  $X$  of real numbers such that  $C_p(X)$  does not have the weak Fréchet–Urysohn property is equal to  $\mathfrak{b}$ .

( $\mathfrak{b}$  is the minimal cardinality of an unbounded family in the Baire space  ${}^{\mathbb{N}}\mathbb{N}$ .) We prove the Kočinac–Scheepers conjecture by showing that if  $C_p(X)$  has the Reznichenko property, then a continuous image of  $X$  cannot be a subbase for a non-feeble filter on  $\mathbb{N}$ .

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## 1. Introduction

A topological space  $X$  has the Fréchet-Urysohn property if for each subset  $A$  of  $X$  and each  $x \in \overline{A}$ , there exists a sequence  $\{a_n\}_{n \in \mathbb{N}}$  of elements of  $A$  which converges to  $x$ . If  $x \notin A$  then we may assume that the elements  $a_n$ ,  $n \in \mathbb{N}$ , are distinct. The following natural generalization of this property was introduced by Reznichenko [7]:

For each subset  $A$  of  $X$  and each element  $x \in \overline{A} \setminus A$ , there exists a countably infinite pairwise disjoint collection  $\mathcal{F}$  of finite subsets of  $A$  such that for each neighborhood  $U$  of  $x$ ,  $U \cap F \neq \emptyset$  for all but finitely many  $F \in \mathcal{F}$ .

In [7] this is called the **weak Fréchet-Urysohn** property. In other works [5, 6, 10] this also appears as the **Reznichenko** property.

For a topological space  $X$  denote by  $C_p(X)$  the space of continuous real-valued functions with the topology of pointwise convergence. A comprehensive duality theory was developed by Arkhangel'skiĭ and others (see, e.g., [2, 9, 5, 6] and references therein) which characterizes topological properties of  $C_p(X)$  for a Tychonoff space  $X$  in terms of covering properties of  $X$ . In [5, 6] this is done for a conjunction of the Reznichenko property and some other classical property (countable strong fan tightness in [5] and countable fan tightness in [6]). According to Sakai [9], a space  $X$  has countable fan tightness if for each  $x \in X$  and each sequence  $\{A_n\}_{n \in \mathbb{N}}$  of subsets of  $X$  with  $x \in \overline{A_n} \setminus A_n$  for each  $n$ , there exist finite sets  $F_n \subseteq A_n$ ,  $n \in \mathbb{N}$ , such that  $x \in \overline{\bigcup_n F_n}$ . In Theorem 19 of [6], Kočinac and Scheepers prove that for a Tychonoff space  $X$ ,  $C_p(X)$  has countable fan tightness as well as Reznichenko's property if, and only if, each finite power of  $X$  has the Hurewicz covering property.

The **Baire space**  ${}^{\mathbb{N}}\mathbb{N}$  of infinite sequences of natural numbers is equipped with the product topology (where the topology of  $\mathbb{N}$  is discrete). A quasiordering  $\leq^*$  is defined on the Baire space  ${}^{\mathbb{N}}\mathbb{N}$  by eventual dominance:

$$f \leq^* g \quad \text{if } f(n) \leq g(n) \text{ for all but finitely many } n.$$

We say that a subset  $Y$  of  ${}^{\mathbb{N}}\mathbb{N}$  is **bounded** if there exists  $g$  in  ${}^{\mathbb{N}}\mathbb{N}$  such that for each  $f \in Y$ ,  $f \leq^* g$ . Otherwise, we say that  $Y$  is **unbounded**.  $\mathfrak{b}$  denotes the minimal cardinality of an unbounded family in  ${}^{\mathbb{N}}\mathbb{N}$ . According to a theorem of Hurewicz [3], a set of reals  $X$  has the Hurewicz property if, and only if, each continuous image of  $X$  in  ${}^{\mathbb{N}}\mathbb{N}$  is bounded. This and the preceding discussion imply that for each set of reals  $X$  of cardinality smaller than  $\mathfrak{b}$ ,  $C_p(X)$  has the

Reznichenko property. Kočinac and Scheepers conclude their paper [5] with the following.

CONJECTURE 1:  $\mathfrak{b}$  is the minimal cardinality of a set  $X$  of real numbers such that  $C_p(X)$  does not have the Reznichenko property.

We prove that this conjecture is true.

## 2. A proof of the Kočinac–Scheepers conjecture

Throughout the paper, when we say that  $\mathcal{U}$  is a **cover** of  $X$  we mean that  $X \subseteq \bigcup \mathcal{U}$  but  $X$  is not contained in any member of  $\mathcal{U}$ . A cover  $\mathcal{U}$  of a space  $X$  is an  $\omega$ -**cover** of  $X$  if each finite subset  $F$  of  $X$  is contained in some member of  $\mathcal{U}$ . This notion is due to Gerlits and Nagy [2], and is starring in [5, 6]. According to [5, 6], a cover  $\mathcal{U}$  of  $X$  is  $\omega$ -**groupable** if there exists a partition  $\mathcal{P}$  of  $\mathcal{U}$  into finite sets such that for each finite  $F \subseteq X$  and all but finitely many  $\mathcal{F} \in \mathcal{P}$ , there exists  $U \in \mathcal{F}$  such that  $F \subseteq U$ . Thus, each  $\omega$ -**groupable** cover is an  $\omega$ -cover and contains a countable  $\omega$ -groupable cover.

In [6] it is proved that if each open  $\omega$ -cover of a set of reals  $X$  is  $\omega$ -groupable and  $C_p(X)$  has countable fan tightness, then  $C_p(X)$  has the Reznichenko property. Recently, Sakai [10] proved that the assumption of countable fan tightness is not needed here. More precisely, say that an open  $\omega$ -cover  $\mathcal{U}$  of  $X$  is  $\omega$ -**shrinkable** if for each  $U \in \mathcal{U}$  there exists a closed subset  $C_U \subseteq U$  such that  $\{C_U : U \in \mathcal{U}\}$  is an  $\omega$ -cover of  $X$ . Then the following duality result holds.

THEOREM 2 (Sakai [10]): For a Tychonoff space  $X$ , the following are equivalent:

- (1)  $C_p(X)$  has the Reznichenko property;
- (2) each  $\omega$ -shrinkable open  $\omega$ -cover of  $X$  is  $\omega$ -groupable.

It is the other direction of this result that we are interested in here. Observe that any **clopen**  $\omega$ -cover is trivially  $\omega$ -shrinkable.

COROLLARY 3: Assume that  $X$  is a Tychonoff space such that  $C_p(X)$  has the Reznichenko property. Then each clopen  $\omega$ -cover of  $X$  is  $\omega$ -groupable.

From now on  $X$  will always denote a set of reals. As all powers of sets of reals are Lindelöf, we may assume that all covers we consider are countable [2]. For conciseness, we introduce some notation. For collections of covers of  $X$ ,  $\mathfrak{U}$  and  $\mathfrak{V}$ , we say that  $X$  satisfies  $\left(\frac{\mathfrak{U}}{\mathfrak{V}}\right)$  (read:  $\mathfrak{U}$  choose  $\mathfrak{V}$ ) if each element of  $\mathfrak{U}$  contains an element of  $\mathfrak{V}$  [14]. Let  $C_\Omega$  and  $C_{\Omega gp}$  denote the collections of clopen  $\omega$ -covers and clopen  $\omega$ -groupable covers of  $X$ , respectively. Corollary 3 says that the Reznichenko property for  $C_p(X)$  implies  $\left(\frac{C_\Omega}{C_{\Omega gp}}\right)$ .

As a warm up towards the real solution, we make the following observation. According to [11], a space  $X$  satisfies  $\text{Split}(\mathfrak{U}, \mathfrak{V})$  if every cover  $\mathcal{U} \in \mathfrak{U}$  can be split into two disjoint subcovers  $\mathcal{V}$  and  $\mathcal{W}$  which contain elements of  $\mathfrak{V}$ . Observe that  $(\overset{C_\Omega}{C_{\Omega^{gp}}})$  implies  $\text{Split}(C_\Omega, C_\Omega)$ . The **critical cardinality** of a property  $\mathbf{P}$  (or collection) of sets of reals,  $\text{non}(\mathbf{P})$ , is the minimal cardinality of a set of reals which does not satisfy this property. Write

$$\mathfrak{rc}_3 = \text{non}(\{X : C_p(X) \text{ has the Reznichenko property}\}).$$

Then we know that  $\mathfrak{b} \leq \mathfrak{rc}_3$ , and the Kočinac–Scheepers conjecture asserts that  $\mathfrak{rc}_3 = \mathfrak{b}$ . By Corollary 3,  $\mathfrak{rc}_3 \leq \text{non}(\text{Split}(C_\Omega, C_\Omega))$ . In [4] it is proved that  $\text{non}(\text{Split}(C_\Omega, C_\Omega)) = \mathfrak{u}$ , where  $\mathfrak{u}$  is the **ultrafilter number** denoting the minimal size of a base for a nonprincipal ultrafilter on  $\mathbb{N}$ . Consequently,  $\mathfrak{rc}_3 \leq \mathfrak{u}$ . It is well known that  $\mathfrak{b} \leq \mathfrak{u}$ , but it is consistent that  $\mathfrak{b} < \mathfrak{u}$ . Thus this does not prove the conjecture. However, this is the approach that we will use: We will use the language of filters to prove that  $\text{non}((\overset{C_\Omega}{C_{\Omega^{gp}}})) = \mathfrak{b}$ . By Corollary 3,  $\mathfrak{b} \leq \mathfrak{rc}_3 \leq \text{non}((\overset{C_\Omega}{C_{\Omega^{gp}}}))$ , so this will suffice.

A **nonprincipal filter** on  $\mathbb{N}$  is a family  $\mathcal{F} \subseteq P(\mathbb{N})$  that contains all cofinite sets but not the empty set, is closed under supersets, and is closed under finite intersections (in particular, all elements of a nonprincipal filter are infinite). A **base**  $\mathcal{B}$  for a nonprincipal filter  $\mathcal{F}$  is a subfamily of  $\mathcal{F}$  such that for each  $A \in \mathcal{F}$  there exists  $B \in \mathcal{B}$  such that  $B \subseteq A$ . If the closure of  $\mathcal{B}$  under finite intersections is a base for a nonprincipal filter  $\mathcal{F}$ , then we say that  $\mathcal{B}$  is a **subbase** for  $\mathcal{F}$ . A family  $\mathcal{V} \subseteq P(\mathbb{N})$  is **centered** if for each finite subset  $\mathcal{A}$  of  $\mathcal{V}$ ,  $\bigcap \mathcal{A}$  is infinite. Thus a subbase  $\mathcal{B}$  for a nonprincipal filter is a centered family such that for each  $n$  there exists  $B \in \mathcal{B}$  with  $n \notin B$ . For a nonprincipal filter  $\mathcal{F}$  on  $\mathbb{N}$  and a finite-to-one function  $f: \mathbb{N} \rightarrow \mathbb{N}$ ,  $f(\mathcal{F}) := \{A \subseteq \mathbb{N} : f^{-1}[A] \in \mathcal{F}\}$  is again a nonprincipal filter on  $\mathbb{N}$ .

A filter  $\mathcal{F}$  is **feeble** if there exists a finite-to-one function  $f$  such that  $f(\mathcal{F})$  consists of only the cofinite sets.  $\mathcal{F}$  is feeble if, and only if, there exists a partition  $\{F_n\}_{n \in \mathbb{N}}$  of  $\mathbb{N}$  into finite sets such that for each  $A \in \mathcal{F}$ ,  $A \cap F_n \neq \emptyset$  for all but finitely many  $n$  (take  $F_n = f^{-1}[\{n\}]$ ). Thus  $\mathcal{B}$  is a subbase for a feeble filter if, and only if:

- (1)  $\mathcal{B}$  is centered,
- (2) for each  $n$  there exists  $B \in \mathcal{B}$  such that  $n \notin B$ , and
- (3) there exists a partition  $\{F_n\}_{n \in \mathbb{N}}$  of  $\mathbb{N}$  into finite sets such that for each  $k$  and each  $A_1, \dots, A_k \in \mathcal{B}$ ,  $A_1 \cap \dots \cap A_k \cap F_n \neq \emptyset$  for all but finitely many  $n$ .

Define a topology on  $P(\mathbb{N})$  by identifying it with **Cantor's space**  ${}^{\mathbb{N}}\{0,1\}$  (which is equipped with the product topology).

**THEOREM 4:** *For a set of reals  $X$ , the following are equivalent:*

- (1)  $X$  satisfies  $(C_{\Omega^{gp}}^{\Omega})$ ;
- (2) for each continuous function  $\Psi: X \rightarrow P(\mathbb{N})$ ,  $\Psi[X]$  is not a subbase for a non-feeble filter on  $\mathbb{N}$ .

*Proof:*  $(1 \Rightarrow 2)$  Assume that  $\Psi: X \rightarrow P(\mathbb{N})$  is continuous and  $\mathcal{B} = \Psi[X]$  is a subbase for a nonprincipal filter  $\mathcal{F}$  on  $\mathbb{N}$ . Consider the (clopen!) subsets  $O_n = \{A \subseteq \mathbb{N} : n \in A\}$ ,  $n \in \mathbb{N}$ , of  $P(\mathbb{N})$ . For each  $n$ , there exists  $B \in \mathcal{B}$  such that  $B \not\subseteq O_n$  ( $n \notin B$ ), thus  $X \not\subseteq \Psi^{-1}[O_n]$ .

As  $\mathcal{B}$  is centered,  $\{O_n\}_{n \in \mathbb{N}}$  is an  $\omega$ -cover of  $\mathcal{B}$ , and therefore  $\{\Psi^{-1}[O_n]\}_{n \in \mathbb{N}}$  is a clopen  $\omega$ -cover of  $X$ . Let  $A \subseteq \mathbb{N}$  be such that the enumeration  $\{\Psi^{-1}[O_n]\}_{n \in A}$  is bijective. Apply  $(C_{\Omega^{gp}}^{\Omega})$  to obtain a partition  $\{F_n\}_{n \in \mathbb{N}}$  of  $A$  into finite sets such that for each finite  $F \subseteq X$ , and all but finitely many  $n$ , there exists  $m \in F_n$  such that  $F \subseteq \Psi^{-1}[O_m]$  (that is,  $\Psi[F] \subseteq O_m$ , or  $\bigcap_{x \in F} \Psi(x) \cap F_n \neq \emptyset$ ). Add to each  $F_n$  an element from  $\mathbb{N} \setminus A$  so that  $\{F_n\}_{n \in \mathbb{N}}$  becomes a partition of  $\mathbb{N}$ . Then the sequence  $\{F_n\}_{n \in \mathbb{N}}$  witnesses that  $\mathcal{B}$  is a subbase for a **feeble** filter.

$(2 \Rightarrow 1)$  Assume that  $\mathcal{U} = \{U_n\}_{n \in \mathbb{N}}$  is a clopen  $\omega$ -cover of  $X$ . Define  $\Psi: X \rightarrow P(\mathbb{N})$  by

$$\Psi(x) = \{n : x \in U_n\}.$$

As  $\mathcal{U}$  is clopen,  $\Psi$  is continuous. As  $\mathcal{U}$  is an  $\omega$ -cover of  $X$ ,  $\mathcal{B} = \Psi[X]$  is centered (see Lemma 2.2 in [13]). For each  $n$  there exists  $x \in X \setminus U_n$ , thus  $n \notin \Psi(x)$ . Therefore  $\mathcal{B}$  is a subbase for a feeble filter. Fix a partition  $\{F_n\}_{n \in \mathbb{N}}$  of  $\mathbb{N}$  into finite sets such that for each  $\Psi(x_1), \dots, \Psi(x_k) \in \mathcal{B}$ ,  $\Psi(x_1) \cap \dots \cap \Psi(x_k) \cap F_n \neq \emptyset$  (that is, there exists  $m \in F_n$  such that  $x_1, \dots, x_k \in U_m$ ) for all but finitely many  $n$ . This shows that  $\mathcal{U}$  is groupable. ■

**COROLLARY 5:**  $\text{non}((C_{\Omega^{gp}}^{\Omega})) = \mathfrak{b}$ .

*Proof:* Every nonprincipal filter on  $\mathbb{N}$  with a (sub)base of cardinality smaller than  $\mathfrak{b}$  is feeble (essentially, [12]), and by an unpublished result of Petr Simon, there exists a non-feeble filter with a (sub)base of cardinality  $\mathfrak{b}$  — see [1] for the proofs. Use Theorem 4. ■

This completes the proof of the Kočinac–Scheepers conjecture.

### 3. Consequences and open problems

Let  $\mathcal{B}_\Omega$  and  $\mathcal{B}_{\Omega^{gp}}$  denote the collections of **countable Borel**  $\omega$ -covers and  $\omega$ -groupable covers of  $X$ , respectively. The same proof as in Theorem 4 shows that the analogue theorem where “continuous” is replaced by “Borel” holds.

$\mathcal{U}$  is a **large** cover of a space  $X$  if each member of  $X$  is contained in infinitely many members of  $\mathcal{U}$ . Let  $\mathcal{B}_\Lambda$ ,  $\Lambda$ , and  $\mathcal{C}_\Lambda$  denote the collections of countable large Borel, open, and clopen covers of  $X$ , respectively. According to [6], a large cover  $\mathcal{U}$  of  $X$  is **groupable** if there exists a partition  $\mathcal{P}$  of  $\mathcal{U}$  into finite sets such that for each  $x \in X$  and all but finitely many  $\mathcal{F} \in \mathcal{P}$ ,  $x \in \bigcup \mathcal{F}$ . Let  $\mathcal{B}_{\Lambda^{gp}}$ ,  $\Lambda^{gp}$ , and  $\mathcal{C}_{\Lambda^{gp}}$  denote the collections of countable groupable Borel, open, and clopen covers of  $X$ , respectively.

**COROLLARY 6:** *The critical cardinalities of the classes  $(\mathcal{B}_\Lambda)$ ,  $(\mathcal{B}_\Omega)$ ,  $(\mathcal{B}_{\Lambda^{gp}})$ ,  $(\mathcal{B}_{\Omega^{gp}})$ ,  $(\Lambda)$ ,  $(\Omega)$ ,  $(\Lambda^{gp})$ ,  $(\Omega^{gp})$ ,  $(\mathcal{C}_\Lambda)$ ,  $(\mathcal{C}_\Omega)$ , and  $(\mathcal{C}_{\Lambda^{gp}})$  are all equal to  $\mathfrak{b}$ .*

*Proof:* By the Borel version of Theorem 4,  $\text{non}((\mathcal{B}_{\Omega^{gp}})) = \mathfrak{b}$ . In [15] it is proved that  $\text{non}((\mathcal{B}_{\Lambda^{gp}})) = \mathfrak{b}$ . These two properties imply all the other properties in the list. Now, all properties in the list imply either  $(\mathcal{C}_{\Lambda^{gp}})$  or  $(\mathcal{C}_{\Lambda^{gp}})$ , whose critical cardinality is  $\mathfrak{b}$  by Theorem 4 and [15]. ■

If we forget about the topology and consider **arbitrary** countable covers, we get the following characterization of  $\mathfrak{b}$ , which extends Theorem 15 of [6] and Corollary 2.7 of [15]. For a cardinal  $\kappa$ , denote by  $\Lambda_\kappa$ ,  $\Omega_\kappa$ ,  $\Lambda_\kappa^{gp}$ , and  $\Omega_\kappa^{gp}$  the collections of countable large covers,  $\omega$ -covers, groupable covers, and  $\omega$ -groupable covers of  $\kappa$ , respectively.

**COROLLARY 7:** *For an infinite cardinal  $\kappa$ , the following are equivalent:*

- (1)  $\kappa < \mathfrak{b}$ ,
- (2)  $(\Lambda_\kappa^{gp})$ ,
- (3)  $(\Omega_\kappa^{gp})$ , and
- (4)  $(\Omega_\kappa^{gp})$ .

It is an open problem [10] whether item (2) in Sakai’s Theorem 2 can be replaced with  $(\Omega_{gp})$  (by the theorem, if  $X$  satisfies  $(\Omega_{gp})$ , then  $C_p(X)$  has the Reznichenko property; the other direction is the unclear one).

For collections  $\mathcal{U}$  and  $\mathcal{V}$  of covers of  $X$ , we say that  $X$  satisfies  $\mathcal{S}_{fin}(\mathcal{U}, \mathcal{V})$  if:

For each sequence  $\{\mathcal{U}_n\}_{n \in \mathbb{N}}$  of members of  $\mathcal{U}$ , there is a sequence  $\{\mathcal{F}_n\}_{n \in \mathbb{N}}$  such that each  $\mathcal{F}_n$  is a finite subset of  $\mathcal{U}_n$ , and  $\bigcup_{n \in \mathbb{N}} \mathcal{F}_n \in \mathcal{V}$ .

In [15] it is proved that  $(\overset{\Lambda}{\Lambda^{gp}}) = S_{fin}(\Lambda, \Lambda^{gp})$  (which is the same as the Hurewicz covering property [6]). We do not know whether the analogous result for  $(\overset{\Omega}{\Omega^{gp}})$  is true.

PROBLEM 8: Does  $(\overset{\Omega}{\Omega^{gp}}) = S_{fin}(\Omega, \Omega^{gp})$ ?

In [6] it is proved that  $X$  satisfies  $S_{fin}(\Omega, \Omega^{gp})$  if, and only if, all finite powers of  $X$  satisfy the Hurewicz covering property  $S_{fin}(\Lambda, \Lambda^{gp})$ , which we now know is the same as  $(\overset{\Lambda}{\Lambda^{gp}})$ .

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